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The Balmorel Model:

Theoretical Background

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1 Introduction

The purpose of the Balmorel project is the construction of a partial equilibrium model covering the electricity and CHP sectors in the countries around the Baltic Sea suited for the analysis of relevant policy questions to the extent that these contain substantial international aspects.

In this paper we give some of the theoretical background for the Balmorel model. The description aims at presenting some basic ideas and properties.

This document is part of a series that together documents the Balmorel model:

Balmorel: A Model for Analyses of the Electricity and CHP Markets in the Baltic Sea Region (Main Report)

The Balmorel Model: Theoretical Background (this document)

The Balmorel Model structure

Balmorel data documentation

Balmorel: Getting started

The present document contains a brief discussion of the relations between the theoretical derivations and the implementation.Observe however that the notation in the present document is not harmonised with the notation in the Balmorel model implementation. This is intentional, because we have here aimed at a more general and mathematically oriented presentation, which is not suited for the names convention, nor for the details of the model structure used in the GAMS language in which the Balmorel is implemented.

Further information, including application examples, may be found at the homepage of the project, www.balmorel.com.

2 The Generation System

2.1 Generation technologies

Generation of heat and electricity - i.e., conversion of other energy forms to heat and electricity - is undertaken in generation units. A generation unit has a number of characteristics, in particular

- a set of feasible combinations of heat (h) and electricity (e) generation
- an efficiency, specifying how much useful energy in the form of heat and/or electricity can be taken out for each unit of primary energy input
- what kind of primary fuels the unit can use
- environmental aspects, specifying how much emission is generated for each unit of primary energy input
- economic aspects, in particular operation and maintenance costs, and investment costs for new capacity.

Further, technological development may take place, this is represented by specifying that some technologies are available only from a certain year, cf. Section 10.

All the characteristics of a unit are represented by a set of linear relations. Thus, for instance for unit i the set of feasible combinations of heat and electricity generation may be specified as

$$g^i(e^i, h^i) \leq 0 \tag{1}$$

where g^i for each i represents a set of linear expressions and e^i and h^i are the electricity and heat generation, respectively.

The emission characteristics may be expressed relative to the generation of electricity and heat, depending e.g. on the individual unit or on the fuel type, or a combination. Thus, emissions of a particular type m (e.g. CO2 or SO2) on unit i (assuming that this also specifies the particular fuel and other necessary characteristics) could be expressed as a linear expression symbolised by

$$\Phi^m(e^i, h^i) \tag{2}$$

Similar linear relations may be specified for the other characteristics. See in particular Section 2.2 for costs, and Section 10 for investment costs.

2.2 Supply function

In the exposition in Section 8 it is assumed that the generation cost function C (the supply function) is explicitly given. However, this will not be the case, and we therefore show how to extend the above result in this respect.

The function C may in a situation with many generation technologies, many producers and perfect competition between them be seen as the supply function derived as follows. An assumption of perfect competition implies that for any total output (e, h) the generation is constituted such that it is done the cheapest possible way, and in the same way as if it had been centrally planned. Therefore for any (e, h) the cost C may be found as

$$C(e, h) = \min_{e^i, h^i} [\sum_{i=1}^{I} C^i(e^i, h^i)] \tag{3}$$

$$g^i(e^i, h^i) \leq 0, \ \forall i \tag{4}$$

$$\sum_{i=1}^{I} e^i = e \tag{5}$$

$$\sum_{i=1}^{I} h^i = h \tag{6}$$

It is here assumed that there are I technologies available, each one defined on the set given by the constraint (4) (identical to (1)) and each one with a cost function C^i. Together these technologies produce the quantities e and h as required in (5) and (6).

If it is assumed that all g^i and C^i are convex then C as defined in (3) - (6) is convex, and it is quite obvious that this derivation of C may be substituted into the problem (66) such that it reads

$$\max_{e,h,e^i,h^i} [U^e(e) + U^h(h) - \sum_{i=1}^{I} C^i(e^i, h^i)] \tag{7}$$

where the optimisation is subject to the constraints (4) - (6), and the optimisation is with respect to variables e, h and e^i, h^i, $i = 1, \ldots, I$.

While the assumptions taken on g^i and C^i indeed assure that C defined in (3) - (6) is convex, C is not necessarily continuously differentiable. This may invalidate (67) - (68). However, this does not invalidate the desired notion of balanced marginal values, it only requires the application of a more general definition of derivative, cf. (69).

3 Geographical Distinctions

The model permits specification of geographically distinct areas. On the supply side the primary reasons for this are related to possibilities of application of and restrictions on generation technologies and resources, to transmission and distribution constraints and costs, and to different national characteristics. On the demand side the reason is the need for specifying different trajectories and elasticities according to consumers' geographically distinct characteristics.

The three basic types of geographical units are areas, regions and countries, and they are related such that a region contains areas while a country contains regions.

As a consequence of this, most exogenous variables will be specified individually, according to the geographical entity, to which they refer. In particular this concerns demand (cf. Section 7), generation technologies (specified for each area, cf. Section 2), electricity transmission (between regions) and distribution (such that electricity distribution is specified by region and heat distribution is specified by area, cf. Section 5).

Further, the endogenous variables will be specified relative to the geographical entities. In particular this concerns generation (Section 2), transmission (Section 5) and consumption (cf. Section 7).

4 Time

The model operates with several time periods. We may distinguish between time periods within the year (i.e., a subdivision of the year) and between the individual years. The latter will be discussed further in Section 10.

Let the year be divided into T time periods, $t = 1, \dots, T$ (further refinement is discussed in Section 4.3). This implies that most exogenous and endogenous variables must be specified with an index t, in particular in relation to generation (Section 2), transmission (Section 5) and consumption (cf. Section 7).

For some basic analyses the sequence of the time segments within a year is of no importance. Hence, the representation of time segments may for such purpose be done equally well by a duration curve representation as by a chronological one. The former may have an advantage in terms of computational efforts upon solution of the model. However, the chronological representation of time may be necessary for modeling of certain components, e.g. whenever intertemporal storage is important, see Section 4.3.

4.1 Demand

It will be assumed that for each time period a demand function can be specified as discussed in Section 7. In particular this implies that there is no substitution between demand in the different time periods, between different geographical units or between heat and electricity demands.

Further, consumption will be found for each time period t.

4.2 Generation

Generation cost functions will have to be specified for each time period t, and generation quantities will also have to be found for each time period.

As concerns investments in new generation technology the following is assumed. At the beginning at the year it is possible to invest in new generation capacity $\overline{e}$ and $\overline{h}$ representing electricity and heat, respectively. This capacity is available from the beginning of the year. Total generation capacity therefore throughout the year consists of the old capacity, existing at the beginning of the year, and the new capacity.

Total generation cost on the old and the new technologies in time period t is given by the functions C_t^{old} and C_t^{new}, respectively. Investments costs for the new technology is C^{inv} (see more on this in Section 10). Generation quantities of electricity and heat on the old and new technologies are e^{old}, h^{old}, e^{new} and

h^{old}, respectively. The total costs of generation and investment during the year will therefore be

$$C^{inv}(\overline{e},\overline{h}) + \sum_{t=1}^{T} C_t^{old}(e_t^{old}, h_t^{old}) + \sum_{t=1}^{T} C_t^{new}(e_t^{new}, h_t^{new}) \tag{8}$$

and the constraints on the generation of the individual units will be

$$g_t^{old}(e_t^{old}, h_t^{old}) \leq 0, \forall t \tag{9}$$
$$g_t^{new}(e_t^{new}, h_t^{new}) \leq 0, \forall t \tag{10}$$
$$e_t^{new} \leq \overline{e}, \forall t \tag{11}$$
$$h_t^{new} \leq \overline{h}, \forall t \tag{12}$$

Optimisation is with respect also to $\overline{e}$ and $\overline{h}$. These variables may in turn be restricted, e.g. because of resource limitations or policy decisions. Further, the expansion of new generation capacities may be more or less constrained.

Observe that in relation to transmission (see Section 5) similar constructions are used, such that at the beginning of a year it is possible to invest in new transmission capacity.

4.3 Storages

Apart from what has already been discussed, the representation of more than one time period per year is necessary for representation of the functioning of some generation technologies, e.g. those with intertemporal storages such as hydro power.

Assume that a certain amount $\overline{w}$ of hydro power is made available at the beginning of the year and let e_t^w be the generation of hydro power during time period t. Then the reservoir balance equation to be fulfilled by the hydro generation technology is

$$\sum_{t=1}^{T} e_t^w \leq \overline{w} \tag{13}$$

Such modeling is valid under the assumptions that there are no limits on the storage capacity nor on the amounts taken in or out of the storage, and no costs depending on the amount stored nor on the amounts taken in or out of the storage.

If such components shall be represented then a more appropriate model could be

$$v_{t+1} = \alpha v_t + \beta^{in} e_t^{in} - \beta^{out} e_t^{out} - \gamma + w_t \tag{14}$$
$$\underline{v} \leq v_t \leq \overline{v} \tag{15}$$
$$\underline{e}^{in} \leq e_t^{in} \leq \overline{e}^{in} \tag{16}$$
$$\underline{e}^{out} \leq e_t^{out} \leq \overline{e}^{out} \tag{17}$$
$$\text{initial and final conditions on } v \tag{18}$$

The modeling may be further refined, but we shall not go into detail with this, see the documentation mentioned on page 5.

5 Transmission and Distribution

As concerns the transmission and distribution characteristics these relate to each of the two products, electricity and heat. The two kinds of products are handled differently, essentially because for a model covering a large geographical region like the Baltic Sea area, transmission of electricity may be seen as being possible while transmission of heat may not. See Figure 1.

The modeling of distribution and transmission is quite simplified, in line with the modeling of the generation system.

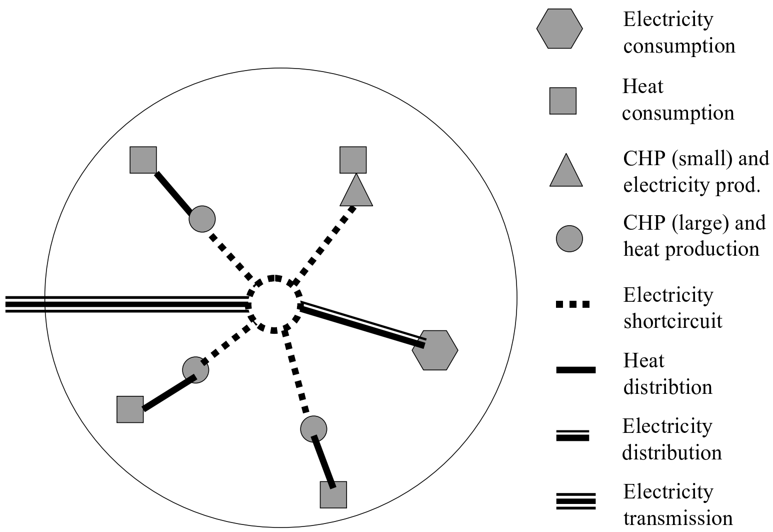

Figure 1: Connections between generation, consumption, distribution and transmission in one region, with one electrical transmission connection to another region.

5.1 Electricity transmission

In order to represent transmission characteristics on electricity the model operates with regions. Within each region the electrical transmission system is considered to be so strongly developed that there are no constraints on the electricity flow. Thus, it may in this respect be considered that in this region all electricity is produced in one node (point) and consumed in another node. Between the generation and consumption nodes there are no constraints, however there is a loss on the electricity flow between the two nodes.

Electricity may be transmitted between the regions or more specifically, between the regions' generation nodes. The transmission implies a loss, proportional to the amount of electricity transmitted such that the region importing electricity is receiving less than that which is exported from the exporting region due to the loss.

Let $x^{(a,b)}$ denote the amount of electricity exported from region j towards region i, and let $\epsilon^{x(j,i)}$ denote the loss related to this transmission. Then the amount of electricity entering the importing region i is

$$x^{(j,i)}(1 - \epsilon^{x(j,i)}) \tag{19}$$

Here, as elsewhere, losses are given as real number in the interval $[0, 1)$.

Further, transmission can not exceed the transmission capacity. The amount transmitted is in this respect considered as that which is sent out of a region. Hence, the formula for the electrical transmission constraint is

$$x^{(j,i)} \leq \hat{x}^{(j,i)} \tag{20}$$

where $x^{(j,i)}$ is the electricity exported and $\hat{x}^{(j,i)}$ is the upper limit on this quantity.

Observe that due to the loss, a transmission constraint must be specific as to whether it refers to the amount of electricity entering or leaving the transmission line; in (20) the former was used.

See Section 4.2 for a description of how the transmission capacity may be increased.

Transmission also implies a cost specified by the coefficient $\beta^{x(j,i)}$, such that transmission cost is $\beta^{x(j,i)}x^{(j,i)}$ for the transmission of $x^{j,i)}$ out of j towards i.

5.2 Electricity exchange with third countries

Electricity may be exchanged with third countries (i.e., countries or electricity regions not otherwise explicitly modeled). This can be handled as fixed quantities (exogenously given). Such model elements can be thought of as generation patterns located in regions in a third countries, if these were included explicitly in the model. Hence, this does not require any modifications of the general setup of the present document.

5.3 Electricity distribution

Within each region there is a loss due to the distribution of electricity from producers (i.e., generation nodes) to consumers. This loss, proportional to the electricity entering the distribution network, is represented by the term ϵ^e. Observe that there are no distribution constraints. The loss implies that

$$e_d = e_s(1 - \epsilon^e) \tag{21}$$

where e_s and e_d are the amounts of electricity arriving at the consumers and entering the distribution network, respectively. (Observe that due to the possibility of transmission, the amount of electricity entering the distribution network in a region need not be identical to electricity generation in that region.)

Distribution also implies a cost β^e, such that distribution cost is $\beta^e e_s$, or, according to (21), $\beta^e e_d/(1 - \epsilon^e)$.

5.4 Heat distribution

Heat demand and heat generation are specified individually for each area, cf. Section 3. Heat transmission between areas is not possible in the model.

Heat distribution in district heating network is possible within any area. This is not subject to any constraints, i.e. it is assumed that a sufficiently strong network exist, in line with the assumptions in relation to the electricity distribution. There is a loss, proportional to the heat generation.

Similarly to the case of electricity distribution the loss implies that

$$h_d = h_s(1 - \epsilon^h) \tag{22}$$

where h_d and h_s are the amounts of heat arriving at the demand end and leaving the supply end (i.e., what is produced), respectively.

Heat distribution also implies a cost β^h, such that distribution cost is $\beta^h h_s$ or, according to (22), $\beta^h h_d/(1 - \epsilon^)$.

5.5 Implications for prices

The costs and losses of distribution will imply that the prices (or more precisely, marginal costs, cf. Section 12.1) of electricity and heat will differ between the producer node and the consumer node in a region. Further, transmission costs and losses will imply that electricity prices will differ between generation nodes in different regions. In Section 12 a broader discussion of prices will be given, here we present preliminary results relative to distribution and transmission.

Thus, let π_d^e and π_d^h denote the prices of electricity and heat, respectively, as perceived by the demand side (however, disregarding taxes, see Section 6), and let similarly π_s^e and π_s^h denote the prices of electricity and heat, respectively, as perceived by the supply side (i.e., producers' sales prices). Let β^e and β^h denote distribution costs of electricity and heat, respectively, and let β^x denote the transmission cost of electricity. All these costs are specified on a per unit basis at the generation node.

Then the following relations are assumed to hold between the electricity prices at a consumer node and at a producer node in the same region:

$$\pi_d^e = (\frac{1}{1-\epsilon^e})(\pi_s^e + \beta^e) \tag{23}$$

and similarly for heat prices

$$\pi_d^h = (\frac{1}{1-\epsilon^h})(\pi_s^h + \beta^h) \tag{24}$$

For electricity transmission the implications are that prices in two generation nodes, where one has import from the other, that prices will be higher in the node that imports. If there is no active transmission constraint between the two nodes, then the relations between the prices will be

$$\pi_i^e = (\frac{1}{1-\epsilon^{x(j,i)}})(\pi_j^e + \beta^{x(j,i)}) \tag{25}$$

where subindexes i and j denote importing and exporting nodes, respectively.

If the transmission constraint is active then this need not hold but rather the following relation holds

$$\pi_i^e \geq (\frac{1}{1-\epsilon^{x(j,i)}})(\pi_j^e + \beta^{x(j,i)}) \tag{26}$$

See further Section 12.

Transmission implies a cost. Thus, if x is the vector of transmission quantities, specified at the exporting nodes, then the associated cost will be $C^x(x)$. C^x will be assumed to be convex.

5.6 Implications for the objective function

With the introduction of distribution and transmission costs the objective function in (64) or (66), with specification as e.g. in (3), should be modified.

We shall consider the distribution and transmission costs as part of the cost side in (64). Hence the function C should be taken to consist of three components, relative to generation, distribution and transmission, respectively. Hence, with the notation introduced above, C is given as

$$C(e_s, h_s, x) = C^g(e_s, h_s) + C^d(e_s, h_s) + C^x(x) \tag{27}$$

Consequently, (64) is modified to

$$\max_{e_d, e_s, h_d, h_s, x} [U^e(e_d) + U^h(h_d) + o + \pi_d^e e_d + \pi_d^h h_d - (C^g(e_s, h_s) + C^d(e_s, h_s) + C^x(x))] \tag{28}$$

subject to the relevant constraints. The budget constraint (65) is similarly with the introduced notation specified to

$$\pi_c^e e_c + \pi_c^h h_c + o = B \tag{29}$$

Hence, (64), (66) or (3) are modified to

$$\max_{e_d, e_s, h_d, h_s, x} [U^e(e_d) + U^h(h_d) - C(e_s, h_s, x)] \tag{30}$$

The expressions may be further specified according to assumptions. Thus, if for instance C^d is assumed linear then with the already given notation C^d may be given as

$$\beta^e e_s + \beta^h h_s \tag{31}$$

6 Taxes

Taxes may be included in various ways. Here, we shall discuss the following types that are implemented

- fuel taxes, for each fuel proportional to the amount consumed in transformation
- emission taxes, for each emission type proportional to the amount emitted
- consumer taxes, proportional to the consumed amount of energy (electricity and heat, respectively)

Other consumer taxes, proportional to the cost of consumed energy (electricity and heat, respectively) are also discussed, see Section 6.4 The introduction of taxes will among other things imply that in relation to the exchange of electricity and heat the amount paid by the consumer will be different from the amount received by the producer.

6.1 Fuel taxes

The addition of a tax on the applied fuel is straightforward. Thus, let two tax rates t_0^f and t_1^f be specified, such that t_0^f is given on a per GJ basis and t_1^f is given on a relative basis. Then if π^f is the fuel price on a per GJ basis without tax, the fuel cost entering the model should be $(\pi^f + t_0^f)t_1^f$ on a per GJ basis.

6.2 Emission taxes

Emission taxes may be implemented in a way similar to that of fuel taxes. Thus, let the emission from a particular combination of fuel and technology be given as $\epsilon^m f$ where f is the amount of fuel and ϵ^m is a constant. Thus, with an emission tax of t^m and a fuel price of π^f the fuel cost entering the model should be $\pi^f + t^m \epsilon^m$.

6.3 Consumer energy taxes

We consider taxes t^e and t^h on electricity and heat, respectively, such that a consumer buying e_d units of electricity at the price π_d^e (excluding taxes) and h_d units of heat at the price π_d^h (excluding taxes) pays $(\pi_d^e + t^e)e_d + (\pi_d^h + t^h)h_d$ for this.

This cost should appear in (65). Observing that (65) should be specified with respect to consumer quantities and prices, it is seen that it then reads

$$(\pi_d^e + t^e)e_d + (\pi_d^h + t^h)h_d + o = B \tag{32}$$

Now (64), (66) or (3) are modified to

$$\max_{e_d, e_s, h_d, h_s, x} [U^e(e_d) + U^h(h_d) - t^e e_d - t^h h_d - C(e_s, h_s, x)] \tag{33}$$

6.4 Value added taxes

As concerns a value added tax with rate v on energy consumption in addition to the above tax this would similarly transform (65) into

$$(1+v)((\pi_d^e + t^e)e_d + (\pi_d^h + t^h)h_d) + o = B \tag{34}$$

However, in this case an attempt to eliminate both the prices and the term o in (64) or (66) will not be successful.

The reason for these limitations may also be discussed in relation to the prices. If these were known it would indeed be possible to operate with the tax as in (34).

These prices could e.g. be specified as being equal to marginal generation costs such that the following two constraints were added to (64) - (65):

$$\pi_s^e = \frac{\partial C(e,h)}{\partial e} \tag{35}$$

$$\pi_s^h = \frac{\partial C(e,h)}{\partial h} \tag{36}$$

where the prices (π_s^e, π_s^h) and (π_d^e, π_d^h) are connected otherwise, e.g. due to the transmission and distribution, cf. Section 5. However, this possibility is not implemented.

It will be possible to include a value added tax v if this applies to all goods. In this case (34) would become

$$(1+v)((\pi_d^e + t^e)e_d + (\pi_d^h + t^h)h_d + o) = B \tag{37}$$

and the situation is similar to the one in (32) except that the budget B is reduced to $B/(1+v)$, i.e., (34) is similar to

$$(\pi_d^e + t^e)e_d + (\pi_d^h + t^h)h_d + o = B/(1+v) \tag{38}$$

It is obvious that this will not change the optimal solution values of e and h, although o will be influenced.

This illustrates that in relation to the essential variables in the model, viz., e and h, the constraint (65) is not really significant. Hence, (65) does not reflect the influence of the budget on e and h. Therefore any attempt to model changes of the budget should be done indirectly, in particular through the functions U^e and U^h in (63).

7 Demands

In this section we investigate various standard functions for describing elastic demands. We shall describe the mathematical form of demand functions, and some of the major consequences of the functional form, in particular the own price elasticity, the cross price elasticity and the income elasticity.

The point of departure is the following formulation:

$$\max[U(Y_1, ..., Y_n)] \tag{39}$$

$$\sum_{i=1}^{n} \pi_i Y_i = B \tag{40}$$

where $U : R^n \to R$ is the consumers' utility function, Y_i is the consumption of good i, π_i is the price of good i and B is the consumers' budget.

In general terms, the solution may, under suitable assumptions implying uniqueness, be written for good i as

$$Y_i^* = Y_i^*(\pi_1, ..., \pi_n; B) \tag{41}$$

i.e., depending of the prices of the individual goods and the consumers' budget, in addition, of course, to the specific form of the function U. Therefore, once (41) is obtained, (39) - (40) are in a sense obsolete.

In the sequel this will be made explicit for different standard forms of U.

7.1 The Cobb-Douglas function, CD

The n-dimensional Cobb-Douglas function may be written as

$$\prod_{i=1}^{n} Y_i^{\alpha_i} \tag{42}$$

where α_i are positive constants. It is often assumed that $\sum_{i=1}^{n} \alpha_i = 1$.

Taking this as the utility function in (39), the consumers' problem of maximising utility under the budget constraint (40) may be solved as follows. The expression in (39) may be substituted by $\max[\sum_{i=1}^{n} \alpha_i \ln Y_i]$ without changing the optimal solution. Introducing a Lagrangian multiplier μ to (40), the problem (39) - (40) may be written

$$\max[\sum_{i=1}^{n} \alpha_i \ln Y_i - \mu(\sum_{i=1}^{n} \pi_i Y_i - B)] \tag{43}$$

First order optimality conditions are

$$\frac{\partial(\sum_{i=1}^{n} \alpha_i \ln Y_i - \mu(\sum_{i=1}^{n} \pi_i Y_i - B))}{\partial Y_i} = \frac{\alpha_i}{Y_i} - \mu\pi_i = 0 \tag{44}$$

or similarly

$$Y_i = \frac{\alpha_i}{\mu\pi_i} \tag{45}$$

Introducing this into (40) yields

$$\sum_{i=1}^{n} \pi_i \frac{\alpha_i}{\mu\pi_i} = B \tag{46}$$

implying

$$\mu = \frac{\sum_{i=1}^{n} \alpha_i}{B} \tag{47}$$

which again, using (45) and assuming $\sum_{i=1}^{n} \alpha_i = 1$, yields

$$Y_i = \frac{\alpha_i B}{\pi_i} \tag{48}$$

This is the so called uncompensated (or Marshallian) demand, the name indicating that the change of prices is not associated with a simultaneous change in the budget B.

Characteristic properties of demand functions are the own price elasticity, the cross price elasticity and the income elasticity.

The own price elasticity, i.e., the relative increase in demand for a relative (percentage) increase in the price may from this be found as

$$\frac{\frac{\partial Y_i}{Y_i}}{\frac{\partial \pi_i}{\pi_i}} = \frac{\partial Y_i}{\partial \pi_i}\frac{\pi_i}{Y_i} = -1 \tag{49}$$

That is, an increase of $x\%$ in the price π_i implies an approximate decrease of $x\%$ in Y_i.

The cross price elasticity, i.e., the relative (percentage) increase in demand of good i for a relative increase in the price of another good j is found as

$$\frac{\frac{\partial Y_i}{Y_i}}{\frac{\partial \pi_j}{\pi_j}} = \frac{\partial Y_i}{\partial \pi_j}\frac{\pi_j}{Y_i} = 0 \tag{50}$$

As seen, the demand for good i is independent of the prices of other goods j, $j \neq i$.

Finally, the income elasticity, i.e., the relative increase in demand for a relative increase in the income (or budget B in (65)) is found as

$$\frac{\frac{\partial Y_i}{Y_i}}{\frac{\partial B}{B}} = \frac{\partial Y_i}{\partial B}\frac{B}{Y_i} = 1 \tag{51}$$

That is, an increase of $x\%$ in the income B implies an approximate increase of $x\%$ in Y_i.

7.2 The Constant Elasticity of Substitution function, CES

The n-dimensional Constant Elasticity of Substitution function, CES, may be written as

$$(\sum_{i=1}^{n} \alpha_i Y_i^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)} \tag{52}$$

where α_i are positive constants. It is often assumed that $\sum_{i=1}^{n} \alpha_i^{\sigma/(\sigma-1)} = 1$.

Assuming $\sigma/(\sigma-1) < 1$ and $\alpha_i > 0$ the function is concave in $Y_i^{(\sigma-1)/\sigma}$ and the expression in (39) may be substituted by $\max[\sum_{i=1}^{n} \alpha_i Y_i^{(\sigma-1)/\sigma}]$. Applying the same technique as above the expression similar to (43) may be written as

$$\max[\sum_{i=1}^{n} \alpha_i Y_i^{(\sigma-1)/\sigma} - \mu(\sum_{i=1}^{n} Y_i - B)] \tag{53}$$

with first order conditions

$$\frac{\partial(\sum_{i=1}^{n} \alpha_i Y_i^{(\sigma-1)/\sigma} - \mu(\sum_{i=1}^{n} \pi_i Y_i - B))}{\partial Y_i} = \alpha_i(\frac{\sigma-1}{\sigma})Y_i^{-\frac{1}{\sigma}} - \mu\pi_i = 0 \tag{54}$$

or

$$Y_i = \frac{1}{\mu^\sigma}(\frac{\alpha_i}{\pi_i})^\sigma(\frac{\sigma-1}{\sigma})^\sigma \tag{55}$$

Inserting this into (40) yields

$$\frac{1}{\mu^\sigma} = \frac{B}{\sum_{i=1}^{n} \alpha_i^\sigma \pi_i^{1-\sigma}}(\frac{\sigma}{1-\sigma})^\sigma \tag{56}$$

Introducing this into (55) gives

$$Y_i = B(\frac{\alpha_i}{\pi_i})^\sigma \frac{1}{\sum_{i=1}^{n} \alpha_i^\sigma \pi_i^{1-\sigma}} \tag{57}$$

Then also for the CES function an explicit expression for the uncompensated demand has been derived. It is seen that the demand for good i as expressed in (57) will depend on the prices of all other goods j, $j \neq i$, in contrast to (48).

The own price elasticity, the cross price elasticity and the income elasticity may then be found as

$$\frac{\frac{\partial Y_i}{Y_i}}{\frac{\partial \pi_i}{\pi_i}} = -\sigma + (1-\sigma)\frac{\alpha_i^\sigma \pi_i^{1-\sigma}}{\sum_{i=1}^{n} \alpha_i^\sigma \pi_i^{1-\sigma}} \tag{58}$$

$$\frac{\frac{\partial Y_i}{Y_i}}{\frac{\partial \pi_j}{\pi_j}} = (\sigma-1)\frac{\alpha_j^\sigma \pi_j^{1-\sigma}}{\sum_{i=1}^{n} \alpha_i^\sigma \pi_i^{1-\sigma}} \tag{59}$$

$$\frac{\frac{\partial Y_i}{Y_i}}{\frac{\partial B}{B}} = 1 \tag{60}$$

Observing that $\pi_i Y_i$ and $\pi_j Y_j$ are the spending on good i and j, respectively, out of total budget B, the last expressions in (58) and (59) may (using (40) and (57)) be interpreted as the budget shares of good i and j, respectively.

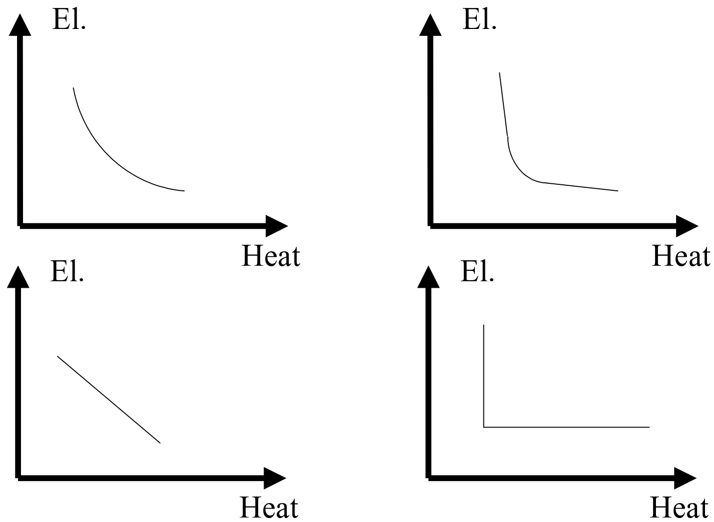

Figure 2: Substitution possibilities in various standard functions related to CES.

7.3 Other standard functions

Other standard functions may be used. Thus, elasticities may be specified by nested CES functions, or LES, Leontief, Stone-Geary or other functions that are commonly used.

In Figure 2 some classical illustrations of substitution are indicated. The upper figures show CES functions, the one on the left indicating a large substitution possibility the other a low subsitution possibility. The bottom left figure indicates the case where substitution is infinite. This case may be derived as a special case of the CES function with $\sigma = \infty$. The bottom right figure indicates the Leontief function where there is no substitution possibilities. Also this may be derived as a special case of the CES function, now with $\sigma = 0$.

7.4 Substitution between electricity and heat

Now consider the introduction of electric heating, viz., the possibility to convert electricity to heat for heating purposes. Assume two techniques available, direct conversion (implying a one-to-one relationship between the electricity used and the heat generated) and heat pumps (where one unit of electricity will generate e.g. 3 units of heat).

Introducing this into the Leontief representation will result in Figure 3. In this figure the slopes (-1 and - 0.33, respectively) of the two middle segments correspond to the two energy conversion efficiencies and the lengths (measured vertically and horisontally, respectively) correspond to the quantities that may be converted.

For high electricity prices the point (H_a, E_a) is relevant, where no conversion from electricity to heat takes place. As the electricity price decreases from such high value, application of heat pumps becomes attractive, which makes the point (H_b, E_b) (full use of heat pumps) relevant. If the electricity price gets even lower, also direct conversion of electricity to heat becomes relevant, with full exploitation of this represented in the point (H_c, E_c).

The modeling of the balances of electricity and heat may be illustrated as follows, disregarding losses in distribution:

$$\sum_{g \in G} e_g - \sum_{j \in J} e_j \quad = \quad e \tag{61}$$

Figure 3: Substitution possibilities with electricity to heat conversion.

$$\sum_{g \in G} h_g + \sum_{j \in J} \eta_j e_j \quad = \quad h \tag{62}$$

Here G is the set of generation technologies (excluding electricity to heat conversion) and J is the set of electricity to heat conversion technologies. η_j is the efficiency of electricity to heat conversion technology j.

7.5 Implementation

As discussed in Section 8 the specification of the demand may conveniently be done in relation to a utility function that is additively separable with respect to electricity and heat and linear with respect to other goods.

Hence, for representing the own price elasticities the investigation of standard functions showed that (at least) the Cobb-Douglass function may be used as a schematic for construction of demand functions. Other properties like cross price elasticity and income elasticity must be represented exogenously (although certain substitutions between heat and electricity may be represented as indicated in Figure 3).

However, there is no reason why the modeling of the utility function should be restricted to a standard function. Thus, any concave, increasing (i.e. non-decreasing) function may be chosen. Or, in relation to the demand function (demand as dependent on the price), any decreasing (i.e. non-increasing) function (for electricity and heat, independently) may be used.

On the other hand, since data may be given in relation to standard forms such as e.g. a Cobb-Douglass or a CES function, the own price elasticities of such function should (and can) then be approximated in the implementation.

Given that the model is linear (i.e., piecewise linear and convex) this may be done as specified in Section 11.3.

8 The Objective Function and Equilibrium Conditions

In this section we derive the basic idea for the specification of equilibrium conditions. The assumptions taken are mainly: two products (electricity and heat), many producers, many technologies, one time period (a static model), one geographical area, one country, no transmission or distribution costs or constraints, smooth (once continuously differentiable) functions, one consumer, short term conditions. These assumptions are discussed and to some extent relieved in other sections. In this section we go for the basic idea.

Assume therefore that the consumer has a utility function $U : R^3 \rightarrow R$ (i.e., a real function with three real arguments) where the arguments are (e, h, o), i.e., electricity, heat and "other". It will be assumed that this is additively separable and quasilinear, such that

$$U(e, h, o) = U^e(e) + U^h(h) + o \tag{63}$$

where $U^e : R \rightarrow R$ and $U^h : R \rightarrow R$. It will be assumed that U^e and U^h are concave, and, as seen from (63), U is linear with respect to o.

We assume that there is one consumer with a total budget of B. Electricity and heat are exchanged at the market at prices π^e and π^h, respectively.

Consider the following problem:

$$\max_{e,h,o}[U(e, h, o) + \pi^e e + \pi^h h - C(e, h)] \tag{64}$$

$$\pi^e e + \pi^h h + o = B \tag{65}$$

In this, $C : R^2 \rightarrow R$ represents generation cost of electricity and heat; it is assumed that C is convex. Further, (65) expresses the consumer's budget restriction.

In (64) the terms $(\pi^e e + \pi^h h - C(e, h))$ may be interpreted as the producer's surplus or utility. The problem (64) - (65) therefore expresses the maximisation of the sum of producer's and consumer's utilities subject to the consumer's budget restriction.

Using (63) and (65), and observing that a constant term (viz., B in (65)) may be eliminated from the objective function without influencing the optimal solution, the problem (64) - (65) may be restated as

$$\max_{e,h}[U^e(e) + U^h(h) - C(e, h)] \tag{66}$$

If it is assumed that the functions are continuously differentiable it is seen that necessary optimality conditions for the problem (66) are

$$\nabla U^e(e) = \frac{\mathrm{d}U^e(e)}{\mathrm{d}e} = \frac{\partial C(e, h)}{\partial e} \tag{67}$$

$$\nabla U^h(h) = \frac{\mathrm{d}U^h(h)}{\mathrm{d}h} = \frac{\partial C(e, h)}{\partial h} \tag{68}$$

Since the functions were assumed concave and convex, respectively, these conditions are also sufficient optimality conditions.

For non-smooth functions the generalised gradient ∂, may be used and (67) - (68) may be reformulated to

$$0 \in \partial(U^e(e) + U^h(h) - C(e, h)) \tag{69}$$

Alternatively the desired notion of balanced marginal values is expressed in relation to the problem (3) - (7), in terms of the Karush-Kuhn-Tucker conditions of optimisation. This will be further discussed in Section 11.

As seen, the conditions state that at the optimum the marginal values of the utility function are equal to the marginal generation costs (partial derivatives) of the two goods e and h. This is precisely the kind of result we want.

Observe that we would arrive at conditions (67) - (68) also by starting directly from (66). However, we found (65) conceptually appealing although the importance of this constraint is otherwise limited, see further Section 6.4.

The result was derived without restrictions on the sign of the variables e, h or o. However, in reality we want these to be positive. This could be achieved by imposing such constraints on the variables in (64) - (65) or in (66). However, it is quite obvious that in the reality which we are modeling, such explicit constraints are not relevant, since they should be automatically fulfilled if the model is otherwise sound. We will therefore prefer to achieve the result by appropriate selection of the functions U^e, U^h and C involved. That this will indeed be possible (at the cost of working with nonsmooth functions) can be demonstrated rigourously, however, we omit this here; see Section 11.

A consequence of the assumed form of the function U in (63) is that it may be recovered by integration of the partial derivatives. Hence,

$$U(e,h,o) = U(0,0,0) + \int_0^e (\nabla U^e(\epsilon))\mathrm{d}\epsilon + \int_0^h (\nabla U^h(\eta))\mathrm{d}\eta + o \qquad (70)$$

Implications

The implications of using (66), or generalisations like (69), are wider than the relations (67) - (68) between the consumers and the producers.

First, as seen from the producers' component in (66), specified in (3) - (6), a balance will be attained between heat and electricity, even if the consumers' utility function is specified as separable. The balance will be with respect to quantities as well as prices.

Second, the year is subdivided into time segments, and some generation technologies, in particular hydro storages, imply a linkage between the segments. Therefore an equilibrium between electricity prices in the various time segments will be attained, as far as the physical constraints (storage and transmission capacities) permit. As just pointed out, this may have implications for the heat side as well.

Third, the long term aspects, covering future years, is represented by the possibility to perform investments. Therefore long term marginal costs may become price setting in periods with capacity shortage. In periods with sufficient capacity the short term marginal costs prevail.

Fourth, the subdivision of the geographical area represented in the model implies that transmission of electricity will take place according to the physical possibilities and economic conditions specified. This will imply equilibrium relations between geographical regions according to the transmission conditions specified.

Thus, the equilibrium conditions cover the two types of agents (producers and consumers), the two products (electricity and heat), the various relevant geographical entities (electrically divided regions), and the various temporal entities (short terms (within the year) and long terms (between years)).

9 Total Model - One year

In the above presentation a number of aspects have been discussed individually. In this section we shall combine the various aspects. We still keep to a quite general formulation while the linear programming implementation will be presented in Section 11. The purpose here is to show that bringing the individual aspects together will not violate the conclusions derived for each of the aspects individually. An additional purpose is to present the general structure of the model in sufficient detail to permit the derivation of the general interpretations to be presented in Section 12. The presentation given in this section is limited to modeling of one year, while dynamic aspects are treated in Section 10.

As concerns the objective function this is specified with components for each of the relevant geographical entities. That is, the consumers' electricity components U^e, $t^e e_d$ should be specified for all regions, and similarly the consumers' heat components U^h, $t^h h_d$ should be specified for all areas. Further, generation components $C(e_s, h_s)$ should be specified for all areas. All distribution costs $d^e e_s$ and $d^h h_s$ should be specified on the relevant regions and areas, respectively, and all transmission costs should be specified for all pairs of regions. Finally, emission costs should be specified for each country.

As concerns the temporal dimension, components like demand and generation shall be specified according to each subinterval of the year, while components like tax rates or fuel prices may be more reasonable represented by values that are assumed constant throughout the year.

Let the following sets be given in the model

- C: the countries, with elements c
- R: the regions, with subsets
  - $R(c)$: the regions in country c
- A: the areas, with subsets
  - $A(c)$: the areas in country c
  - $A(r)$: the areas in region r
- G: the generation units, with subsets
  - $G(c)$: the set of generation units in country c
  - $G(r)$: the set of generation units in region r
  - $G(a)$: the set of generation units in area a
- T: the time periods within the year
- M: the types of emission

Indexes

- a: area
- r: region
- c: country
- t: time period within the year
- s: supply
- d: demand
- m: emission type

Let the variables in the model be

- $e^t_{s,g}$: electricity generation on unit g in period t
- $h^t_{s,g}$: heat generation on unit g in period t
- $e^{r,t}_d$: electricity demand on region r in period t
- $h^{a,t}_d$: heat demand on area a in period t

Let the functions in the model be

- K^t_g: Generation cost on unit g during time period t
- $X^{x(r,\rho)}$: investment cost for transmission capacity
- g^t_g: technical constraint on unit g time period t

- Φ^m: emission of type m

Then the principles of the total model may be specified as follows.

The criterion function may be specified as

$$\begin{gathered}\max[\sum_{t\in T}\{\sum_{c\in C}\{\sum_{r\in R(c)} U^{e,r,t}(e_d^{r,t}) \\ +\sum_{a\in A(c)} U^{h,a,t}(h_d^{a,t}) \\ -\sum_{r\in R(c)} t^e e_s^{r,t}(1-\epsilon_r^e) - \sum_{a\in A(c)} t^h h_s^{a,t}(1-\epsilon_r^h) \\ -\sum_{p\in A(c)} K_p^t(e_{s,p}^t, h_{s,p}^t) \\ -\sum_{(\rho,r):\rho\in R(c),\rho\neq r} \beta^{x(r,\rho)} x^{(r,\rho)} - \sum_{(\rho,r):\rho\in R(c),\rho\neq r} X^{x(r,\rho)} \\ -\sum_{r\in R(c)} \beta_r^e e_d^t/(1-\epsilon_r^e) - \sum_{a\in A(c)} \beta_a^h h_d^t/(1-\epsilon_a^h)\}\}]\end{gathered} \tag{71}$$

Here the terms in the first and second lines are recognised as the consumers' utilities, cf. e.g. (66). The third line represents consumers' energy taxes, cf. Section 6.3, but possibly also other taxes, (cf. Section 6.3 and Section 6.4). The fourth line represents generation costs, cf. (1), including any fuel and emission taxes, cf. Section 6.1 and 6.2, and also including operations and maintenance costs and investment costs, cf. Section 4.2. The fifth line represents transmission operations and investments costs, cf. Section 5.1 (the notation is simplified, but it is understood that double counting should be prevented), and the sixth line represents distribution costs, cf. Section 5.3 and Section 5.4.

The electricity balances are given as

$$\begin{gathered}\sum_{g\in G(r)} e_{s,g}^t + \sum_{\rho\in R,\rho\neq r} x^{(\rho,r),t}(1-\epsilon^{x(\rho,r)}) = \\ e_d^{r,t}/(1-\epsilon_r^e),\ \forall r\in R, \forall t\in T\end{gathered} \tag{72}$$

The heat balances are given as

$$\sum_{g\in G(a)} h_{s,g}^t = h_d^{a,t}/(1-\epsilon_r^h),\ \forall a\in A, \forall t\in T \tag{73}$$

Restriction on individual generation units, including those relating to investments, cf. e.g. (11) and (12), are given by

$$g_g^t(e_{s,g}^t, h_{s,g}^t) \le 0,\ \forall g\in G, \forall t\in T \tag{74}$$

Upper limits on emissions may be given e.g. as follows, where it is assumed that a maximal annual amount is specified for each country

$$\sum_{g\in G(c)}\sum_{t\in T} \Phi^m(e_{s,g}^t, h_{s,g}^t) \le \overline{m}_c,\ \forall c\in C, \forall m\in M \tag{75}$$

In addition to that listed above in (72) - (75), there may be lower and upper bounds on the individual variables, as e.g. (11), (12), (20), or further refinements as described e.g. in relation to storage in Section 4.3.

In addition to the above specifications any number of details may be added. This may influence the solution of the model, of course. However, as long as the formulation is kept linear this will not jeopardise the general results given.

10 The dynamics

The model is implemented as a dynamic model as follows. Each year is subdivided into sub periods, as described in Section 4. During the simulation of one year, all time sub periods are considered interdependently, through a simultaneous optimization. The exogenous parameters relative to this are among other things installed generation and transmission capacities at the beginning of the year.

The result of the simulation is a number of physical quantities, including new capacities for generation and transmission installed during the year. These are then transferred to the beginning of the next year.

Hence, the inter annual dynamics of the model may be characterized as *myopic*.

11 Linear Programming Implementation

The model is implemented as a linear programming model.

The advantages of this is that it provides more easy and efficient numerical solution of the model. Moreover, most of the data is not available with an accuracy that justifies nonlinear relations. Finally, the equilibrium conditions, e.g. (69), may be elegantly expressed as the equivalent KKT conditions, see Section 11.4.

The disadvantages are that for certain aspects the linear functions are theoretically inconvenient. This for instance holds true for specification of elastic demands, that traditionally are modelled with continuous and smooth relationships between price and demand. Also upon interpretation of results of the model simulations discontinuous price movements may be found. A number of theoretical results are traditionally derived under assumption of strict convexity (or concavity, as the specific circumstances dictate) of some of the functions.

The linear modeling does guarantee most other of the usually found essential characteristics of theoretical models and also some empirical observations. Thus, although the supply cost function as derived in (3) is not smooth, it is convex (in particular piecewise linear); consequently the marginal cost of supply is non-decreasing and piecewise constant (and hence, at some points, not unique but rather to be given only within an interval). What might be lacking for some theoretical results is thus the strict convexity of the supply cost function and the continuity of the marginal cost function. The linear functions do not guarantee a unique solution to the model, nor, as just pointed out, unique prices. It is believed that this will only exceptionally be of any importance.

11.1 The Generation System

The generation system is modelled by linear relations as already specified, see Section 2.

11.2 Transmission and Distribution

Transmission and distribution are modelled by linear relations as already specified, see Section 5.

11.3 Demands

The demand structure is specified as follows.

Consider electricity demand in one region and for one time period.

From the previous discussions in Section 8 and Section 7 it follows that we should choose a decreasing function for demand as depending on the prices.

Given that the model is linear (i.e., piecewise linear and convex) this may be done as follows. For the demand we first specify a nominal demand (that which might be spcified in a model with inelastic demands). On top of this we specify deviation steps, that link price steps with steps for changes in demand.

Thus, in a model with inelastic demands the balance between generation and demand might be written

$$\sum_i e_i^t = \hat{D}^t \tag{76}$$

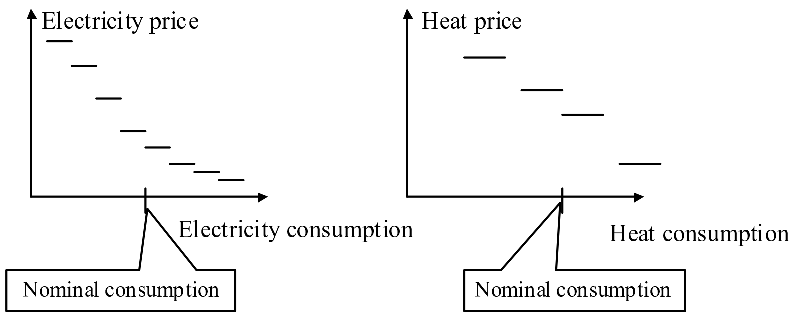

Figure 4: Consumption of electricity and heat depend on the price.

where e_i^t is generation on unit i during time period t, and $\hat{D}^t$ is demand during time period t. Introducing variables u_s^t (representing increased consumption relative to the nominal one, viz., $\hat{D}^t$) and d_r^t (representing decreased consumption relative to $\hat{D}^t$), constrained to $0 \leq u_s^t \leq \overline{u}_s^t$ and $0 \leq d_r^t \leq \overline{d}_r^t$, we substitute (76) by

$$\sum_i e_i^t = \hat{D}^t + \sum_s u_s^t - \sum_r d_r^t \tag{77}$$

The total demand D^t is seen on the right hand side of (77). The size of this will be determined endogenously. This way, the demand is elastic.

It follows that

$$\sum_s d_s^t = \hat{D}^t - D^t \tag{78}$$

$$\sum_r u_r^t = D^t - \hat{D}^t \tag{79}$$

Introducing positive numbers α_r^t and β_s^t, and adding the following terms to the objective function

$$\sum_s \alpha_s^t u_s^t - \sum_r \beta_r^t d_r^t \tag{80}$$

the balancing of demand D^t against price is seen to be achieved. See also Figure 4 and Figure 5.

Assuming $\alpha_{s+1}^t > \alpha_s^t$, $\alpha_1^t > \beta_1^t$, $\beta_r^t < \beta_{r+1}^t$, optimization will assure that either are all $d_s^t = 0$ or all $u_s^t = 0$, and further that if $d_{s+1} > 0$ then $d_s^t = \overline{d}_s^t$, and if $u_{r+1} > 0$ then $u_r^t = \overline{u}_r^t$.

The consumers' marginal utility π_d^t will satisfy

$$\alpha_{r+1}^t \geq \pi_d^t \geq \alpha_r^t \quad \text{if} \quad d_r > 0 \text{ and } d_{r+1} = 0 \tag{81}$$

$$\beta_{s+1}^t \leq \pi_d^t \leq \beta_s^t \quad \text{if} \quad u_s > 0 \text{ and } u_{s+1} = 0 \tag{82}$$

As seen, the demand function is piecewise constant. Apart from this functional form it may be specified arbitrarily, as long as it is non-increasing. Thus, it is indeed quite flexible.

There should be one function specified as in (77) and (80) for heat demand in each heat area and for each time period, and one function for electricity demand in each electricity region and for each time period.

Substitution between electricity and heat is discussed in Section 7.4.

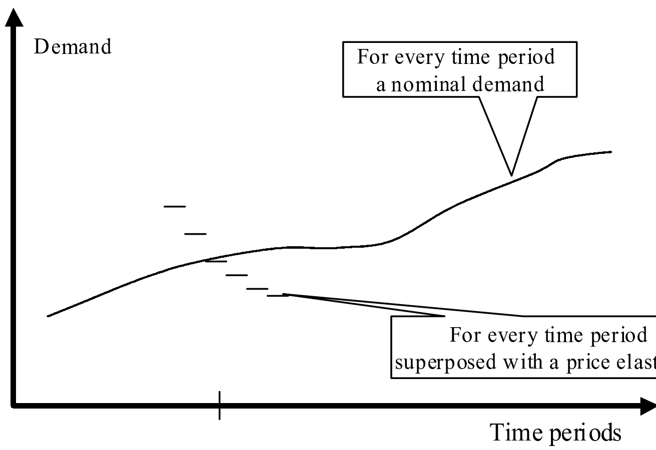

Figure 5: Trajectory of nominal demand (electricity or heat).

11.4 Equilibrium and KKT Conditions

Now consider the equilibrium conditions, e.g. (69). Within the linear formulation an equivalent condition may be stated as follows.

We rewrite the problem (71) - (75) to the following compact form

$$\max_x [f(x)] \tag{83}$$
$$g(x) \leq \gamma \tag{84}$$
$$h(x) = \eta \tag{85}$$

Here, x represents the decision variables in the problem, f, g and h represent the various functions.

If the problem is well defined then there is an optimal solution x^* (although not necessarily unique). Then, as (71) - (75) is a linear programming problem, there exist Lagrange multiplier vectors λ and μ corresponding to (84) and (85), respectively, satisfying

$$\nabla_x f(x^*) = \lambda \nabla_x g(x^*) + \mu \nabla_x h(x^*) \tag{86}$$
$$\lambda \geq 0 \tag{87}$$
$$\lambda(g(x^*) - \gamma) = 0 \tag{88}$$

These conditions, together with the obvious conditions of feasibility of x^* in (84) - (85), are the so called Karush-Kuhn-Tucker (or Kuhn-Tucker) conditions.

As just explained, if an optimal solution x^* exists then there exist Lagrange multiplier vectors λ and μ such that these conditions hold at x^*. However, the argumentation also holds the other way round, i.e., if a (x^*, λ, μ) can be found satisfying (86) - (88) then x^* is optimal in (83) - (85).

11.5 Modeling System Implementation

The model (71) - (75) is implemented in the GAMS modeling language. See further the documents mentioned on page 5. However, the particular modeling language does not influence the theoretical properties of the model: it is still linear. The modeling and solution environment applied ensures among other things that an optimal solution is found along with associated dual variables (Lagrange multipliers), see Section 11.4 and Section 11.6.

We point out that from the information about the solution supplied by GAMS it is not known whether the solution and/or the dual variables are unique.

11.6 Solution Output

The result of the solution effort is an optimal solution and associated Lagrange multipliers. The Lagrange multipliers are often called dual variables, and their optimal values the dual solution, and correspondingly the optimal solution to the problem may be called the primal solution.

The primal solution consist of optimal values of those entities (variables or endogenous variables) in which we in part have formulated the problem. Typically this is generation and demand of electricity and heat, investment in generation capacities, and in transmission capacities, according to the geographical and temporal extension of the problem.

The dual variables are not used in the model formulation, however, they are very useful for the interpretation of the results, as they may be interpreted as prices; sometimes the dual variables are also called shadow prices. See Section 12.

12 Interpretations

The result of the optimization will be optimal values of primal and dual variables, as explained in Section 11.6, such that the equilibrium conditions are fulfilled in the form of the KKT conditions, see Section 11.4.

In general terms it is quite easy to interpret the primal variables, as they enter directly in the model formulation.

More difficulty may be encountered in the interpretation of the dual variables. As is known from optimization theory they can be interpreted as marginal costs or prices, however, it is not always obvious how. We therefore give interpretations of some of the central dual variables in this section.

As concerns the constraints (72) - (75) they are of physical nature, and since they for a well and solved formulated model will be fulfilled, there is not more information to be derived from the optimal solution than is already contained in the formulation.

In contrast to this, the criterion function (71) is not necessarily easy to interpret.

Optimal values of primal and dual variables may be combined to useful measures, in Section 12.4 we discuss in particular consumers' and producers' surplus.

12.1 Marginal costs and shadow prices

The objective function (71) is imperative in the determination of the optimal values of the decision variables in the problem (71) - (75). However, the optimal value of the function in itself may contain limited information, while changes in this optimal values, as driven by changes in some of the parameters defining the problem (71) - (75), can be given important interpretations, as will be derived in this section.

We rewrite the problem (71) - (75) to the following compact form

$$Z^*(\alpha,\gamma,\eta) = \max_x [f(\alpha;x)] \tag{89}$$

$$g(\alpha;x) \leq \gamma \tag{90}$$

$$h(\alpha;x) = \eta \tag{91}$$

which is equivalent to (83) - (85), except that in addition to γ and η also α, that represents other parameters (efficiencies, demands, generation capacities, etc.) in the problem, has been explicitly identified. Also (86) - (88) is similar with this formulation:

$$\nabla_x f(\alpha;x^*) = \lambda \nabla_x g(\alpha;x^*) + \mu \nabla_x h(\alpha;x^*) \tag{92}$$

$$\lambda \geq 0 \tag{93}$$

$$\lambda(g(\alpha;x^*) - \gamma) = 0 \tag{94}$$

It is well known that λ and μ may be interpreted as shadow prices. This means that if $Z^*(\alpha,\gamma,h)$ is continuously differentiable at the particular values (α,γ,η) then

$$\frac{\partial Z^*(\alpha,\gamma,\eta)}{\partial a} = \frac{\partial (f(\alpha;x^*) - \lambda(g(\alpha;x^*) - \gamma) - \mu(h(\alpha;x^*) - \eta))}{\partial a} \tag{95}$$

and similar results hold for γ and η.

The expression (95) (and the similar ones for γ and η) means that the change in Z^* resulting from a change in (α,γ,η) may be estimated, once x^*, λ and μ are known. This latter information is produced by most standard optimization software in the form of dual variables.

Relative to γ and η the expression (95) takes particularly simple forms, viz.,

$$\frac{\partial Z^*(\alpha,\gamma,\eta)}{\partial \gamma} = \lambda \tag{96}$$

$$\frac{\partial Z^*(\alpha,\gamma,\eta)}{\partial \eta} = \mu \tag{97}$$

These two latter expression have the interpretation that a unit change in the right hand side of (84) or (85) lead to an approximate change in Z^* of λ or μ, respectively.

From this the expression "shadow price" may be justified. For instance the decision maker will be willing to "pay" up to μ for one unit increase in η in (85).

In the following sections we shall give some more specific interpretations.

The expressions (95) - (97) presume for validity that Z^* is continuously differentiable at the point (α,γ,η). In general, this is not the case. This is closely related to uniqueness of x^*, λ and μ, since uniqueness of all these three vectors implies continuous differentiability of Z^*. Optimization software does not generally provide information about uniqueness of x^*, λ or μ.

If Z^* is not continuously differentiable then more general expressions may be stated; as explained, this implies substitution of (67) - (68) by (69) (where in turn the latter one is substituted by the KKT conditions (86) - (88)). The practical implication of absence of continuous differentiability is that the shadow values (marginal cost, prices) are not unique.

If a solution exists but the three vectors are not unique then λ may be interpreted as follows. Any value of λ consistent with (86) - (88) is an upper bound on the increase of Z^* for a unit increase on γ, and any value of λ, consistent with (86) - (88) is a lower bound on the decrease of Z^* for a unit decrease on γ. Similar interpretation may be applied to μ and η.

As concerns the units in which the shadow prices are expressed this follows from the units in which the objective function (83) and the corresponding constraints (84) or (85) are expressed. See the sections below for specifications.

Finally observe that reformulation of the objective function or a constraint to a mathematically equivalent form (such that the optimal solution x^* is unaffected) may change the values and interpretations of λ and μ. See Section 12.2 for examples.

12.2 Electricity prices

The equation (72) expresses balance between electricity supply (left hand side) and demand (right hand side) in region r time period t. Let $\mu^t_{r\cdot}$ be the associated Lagrange multiplier. It follows from the discussion in Section 12.1 that $\mu^t_{r\cdot}$ may be interpreted as the marginal cost, or shadow price, of electricity generation in region r time period t.

Obviously the numerical value of the shadow price will depend on the units used in specifying the expressions. Assume in the following that the objective function (71) is expressed in MMoney (where MMoney is millions of e.g. Euro, USD, Mark, Kroner or Litas, etc.) and that the electricity balance equation (72) is expressed in MW.

The unit in which μ_r^t is expressed is millions of money per MW, MMoney/MW. This follows from (71) being expressed in MMoney and (72) being expressed in MW.

To get this in the more familiar unit of Money/MWh, the following should be done. First, μ_r^t should be multiplied by 10^6 to get MMoney converted to Money. Second, the result should be divided by the number of hours for which this equation holds, viz., w^t. Hence, the marginal price of electricity generation in region r time period t is $10^6\mu_r^t/w^t$ Money/MWh.

From the general idea of the model construction, cf. Section 8, it follows that there is equality between producers' marginal utility and the producers' marginal costs, however, appropriately adjusted for the costs and losses of transmission and distribution.

As seen, if the right hand side of (72) is increased by one unit then, according to the interpretation around (97) and the precise formulation of (72), this could be compensated by a unit increase in generation. Therefore $10^6\mu_r^t/w^t$ is the marginal cost of generation.

Now, (72) may be reformulated to the following mathematically equivalent form:

$$
\begin{aligned}
(\sum_{g\in G(r)} e_{s,g}^t + \sum_{\rho\neq r} x^{(\rho,r),t}(1-\epsilon^{x(\rho,r)}))(1-\epsilon_r^e) = \qquad (98)\\
e_d^{r,t},\ \forall r\in R, \forall t\in T
\end{aligned}
$$

This will not change the optimal solution, however, it will change the numerical value of μ_r^t. Consistent with this, the interpretation of $10^6\mu_r^t/w^t$ in relation to (98) is no longer as that of marginal cost of generation, but as the marginal marginal cost of supply at the consumers' locations. (These two meanings will differ if there is a loss or a cost associated with distribution.) Observe that according to (67) the marginal marginal cost of supply at the consumers' locations will be identical to consumers' marginal utility (disregarding consumer taxes).

This illustrates the importance of specificity in the interpretations.

12.3 Heat prices

The interpretation of (73) is quite similar to that of (72) in Section 12.2. Thus, if μ_a^t is the Lagrange multiplier to (73) for area a time period t associated with the optimal solution, then $10^6\mu_a^t/w^t$ is the marginal generation price of heat in area a time period t, expressed in Money/MWh.

12.4 Consumers' and producers' surplus

From knowledge of consumers' utility functions, the producers' marginal cost functions, the amounts consumed and produced and the associated prices it is possible to derive consumers' and producers' surplus as follows.

Consider electricity consumption for a particular time period and a particular electricity demand region. Let the consumed amount of electricity be D and let the consumers' price be π_d^e. With the consumers' utility function specified as in (63) the consumers' surplus may be expressed as

$$
\int_0^D (U^e(e) - \pi_d^e)\mathrm{d}e \qquad (99)
$$

In particular observe that this result depends on the assumed additivity of the utility with respect to electricity and heat. From this, changes in consumers' utility with respect to electricity from a situation characterized by a price and demand pair $(\pi^{e_d^0}, e_d^0)$ to a situation $(\pi_d^{e_d^1}, e^1)$ may be derived as

$$\begin{aligned}
& \int_0^{e_d^1} (U^e(e) - \pi_d^{e_d^1})\mathrm{d}e - \int_0^{e_d^0} (U^e(e) - \pi_d^{e_d^0})\mathrm{d}e \qquad (100) \\
= & \int_0^{e_d^0} (U^e(e) - \pi_d^{e_d^1})\mathrm{d}e + \int_{e_d^0}^{e_d^1} (U^e(e) - \pi_d^{e_d^1})\mathrm{d}e \\
& - \int_0^{e_d^0} (U^e(e) - \pi_d^{e_d^0})\mathrm{d}e \\
= & \int_0^{e_d^0} (\pi_d^{e_d^0} - \pi_d^{e_d^1})\mathrm{d}e + \int_{e_d^0}^{e_d^1} (U^e(e) - \pi_d^{e_d^1})\mathrm{d}e \\
= & (\pi_d^{e_d^0} - \pi_d^{e_d^1})e^0 + (e_d^0 - e_d^1)\pi_d^{e_d^1} + \int_{e_d^0}^{e_d^1} U^e(e)\mathrm{d}e
\end{aligned}$$

Similar to this, the changes in consumers' surplus with respect to heat from a situation characterized by a price and demand pair $(\pi_d^{h^0}, h_d^0)$ to a situation $(\pi_d^{h^1}, h_d^1)$ may be derived as

$$(\pi_d^{h_d^0} - \pi_d^{h_d^1})h^0 + (h_d^0 - h_d^1)\pi_d^{h_d^1} + \int_{h_d^0}^{h_d^1} U^h(h)\mathrm{d}h \qquad (101)$$

Observe in particular, that in order to calculate changes in consumers' surplus it is sufficient to know the utility function U^e in the interval from e_d^0 to e_d^1 and similarly U^h in the interval from h_d^0 to h_d^1.

Total changes in consumers' surplus are calculated by summation over all time periods and all electricity demand regions.

As concerns producers' surplus we may proceed as follows. Consider a specific geographical entity, where there is only one node of electricity and one node of heat. Compare the changes from a situation characterized by a quadruple $(\pi_s^{e^0}, e_s^0, \pi_s^{h^0}, h_s^0)$ to a situation $(\pi_s^{e^1}, e_s^1, \pi_s^{h^1}, h_s^1)$. The change in producers' surplus is then

$$\pi_s^{e^1} e_s^1 + \pi_s^{h^1} h_s^1 - C(e_s^1, h_s^1) - (\pi_s^{e^0} e_s^0 + \pi_s^{h^0} h_s^0 - C(e_s^0, h_s^0)) \qquad (102)$$

where C is the cost of generation, cf. (3).

Total changes in producers' surplus are calculated by summation over all time periods, all electricity generation regions and all heat generation areas.

12.5 Generality

We have here given a few examples of interpretation of model result. The same principles of deriving interpretations may be applied to other relevant model elements. The basis of the interpretation is as explained above the global optimal solution and the associated dual variables. From this the specific interpretations are derived according to the real content.

13 Conclusions

The present document has given the main background elements for the Balmorel model. The main points presented here may be summarised as follows:

- The model represents generation, transportation and consumption of energy.
- The model is formulated as a convex model, or, more specifically, as a linear (i.e., convex and piecewise linear) model.

- The supply system includes electricity and heat, various generation units including CHP, various fuels, variable costs, investment costs, emissions, losses, distribution and transmission costs and losses, transmission limits, fuel taxes, emission taxes and quota and other features.
- Demand is represented by a consumers' utility function with respect to electricity and heat that is assumed to be additively separable between the two goods, and for each of them to be non-increasing with increasing consumption. Substitution possibilities between the two goods are represented otherwise.
- The model has a distinction between geographical units, to represent possibilities and costs in relations to transportation of electricity and heat, and to represent various national and physical differences.
- The model has a resolution of time within the year, permitting e.g. representation of demand variations and intertemporal storage (hydro, heat).
- As concerns the dynamic aspects over several years, the model is solved for one year at the time, with the forward-looking mechanism (having implications for investment decision in that year) represented by knowledge of long term marginal costs of investment.
- The guiding principle for the determination of the endogenous variables is that there should be a balance between consumers' and producers' surplus, as concerns the electricity and heat sectors. In this sense, the model is a partial equilibrium model. The equilibrium conditions cover the two types of agents (producers and consumers), the two products (electricity and heat), the various relevant geographical entities (electrically divided regions), and the various temporal entities (short terms (within the year) and long terms (between years)).
- The equilibrium conditions are within the modeling context translated to mean the KKT conditions.
- The model results include information about physical quantities like electricity and heat generated according to time, space, generation technology and fuel, and derived information that may be interpreted as e.g. marginal cost, or producers' sales prices, or consumers' prices.

We should emphasize that any number of exogenous parameters may be included in the model (as long as these are represented by linear relations) without violating the basic mechanisms at the level of discussion applied in the present paper. Hence all the conclusions derived here hold true for any implementation of data structures, and therefore also for the particular one applied in the Balmorel model, cf. the documents mentioned on page 5.